Global well-posedness to stochastic reaction-diffusion equations on the real line $\mathbb{R}$ with superlinear drifts driven by multiplicative space-time white noise

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Introduction

In this paper, we are concerned with the well-posedness of the following stochastic reaction diffusion equation(SRDE):

$$
\begin{cases}
\mathrm{d}u(t,x) = \frac{1}{2}\Delta u(t,x)\,\mathrm{d}t + b(u(t,x))\,\mathrm{d}t \\
\qquad\qquad\quad + \sigma(u(t,x))\,W(\mathrm{d}t,\mathrm{d}x), \;\; t>0, \;\; x\in\mathbb{R}, \\
u(0,x) = u_0(x), \quad x\in\mathbb{R}.
\end{cases}
\tag{2.1}
$$

- $b, \sigma : \mathbb{R} \to \mathbb{R}$ deterministic measurable functions.
- drift b is locally Log-Lipschitz and $|b(z)| = O(|z|\log|z|)$.
- $W:$ space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$ defined on some filtrated probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

Introduction

- Numerous work in the literature on stochastic reaction-diffusion equations driven by space-time white noise. The majority of the work are focused on stochastic reaction-diffusion equations defined on finite intervals instead of the whole real line $\mathbb{R}$, partly due to the essential difficulties brought by the non-compactness of the whole space.
- [S] : the existence and uniqueness of solutions of stochastic reaction-diffusion equations (SRDEs) on $\mathbb{R}$ under the Lipschitz conditions of the coefficients
- [MP] and [MPS] : pathwise uniqueness for stochastic reaction-diffusion equations on $\mathbb{R}$ with Hölder continuous coefficients.

Introduction

- SPDEs with locally Lipschitz coefficients that have polynomial growth and/or satisfy certain monotonicity conditions. The typical example of such a coefficient is $b(u) = -u^3$, which has the effect of "pulling the solution back toward the origin."
- [DKZ](AOP,2019) : global well-posedness of SRDEs on finite intervals, the coefficients are locally Lipschitz and of $(|z| \log |z|)$-growth. Unfortunately, the methods are not valid for SRDEs on $\mathbb{R}$ because typically $||u(t)||_\infty = \sup_{x \in \mathbb{R}} |u(t,x)| = \infty$.

Introduction

- Our results: global well-posedness when the drift b is locally Log-Lipschitz and $|b(z)| = O(|z| \log |z|)$.
- We are forced to work on $C_{tem}(\mathbb{R})$ with a specially designed norm

$$\sup_{t \le T, x \in \mathbb{R}} \left(|u(t,x)| e^{-\lambda |x| e^{\beta t}} \right).$$

- Establish some new, precise (lower order) moment estimates of stochastic convolution on $\mathbb{R}$ and hence obtain some a priori estimates of the solution.
- Pathwise uniqueness: we are not able to apply the usual localization procedure as in the literature. We provide a new type of Gronwall's inequalities, which is of independent interest.

Statement of the main results

Definition of solutions 1

A random field solution to equation (2.1) is a jointly measurable and adapted space-time process $u := \{u(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ *such that for every* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\begin{aligned} u(t,x) =& P_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x,y) b(u(s,y)) \, \mathrm{d}s\mathrm{d}y \\ &+ \int_0^t \int_{\mathbb{R}} p_{t-s} \sigma(u(s,y)) \, W(\mathrm{d}s, \mathrm{d}y), \quad \mathbb{P}-a.s., \end{aligned} \tag{3.1}$$

where $p_t(x,y) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$, *and* $\{P_t\}_{t \geq 0}$ *is the corresponding heat semigroup on* $\mathbb{R}$.

Remark 2

mild solution $\Longleftrightarrow$ *weak solution, in the sense of PDEs*

C_{tem} space

$$C_{tem} := \left\{ f \in C(\mathbb{R}) : \sup_{x\in\mathbb{R}} |f(x)| e^{-\lambda|x|} < \infty \text{ for any } \lambda > 0 \right\},$$

endow it with the metric: for any $f, g \in C_{tem}$,

$$d(f,g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \min\left\{ 1, \sup_{x\in\mathbb{R}} |f(x) - g(x)| e^{-\frac{1}{n}|x|} \right\}.$$

- $f_n \to f$ in C_{tem} iff $\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| e^{-\lambda|x|} \to 0$ as $n \to \infty$ for any $\lambda > 0$,
- (C_{tem}, d) is a Polish space.

Hypotheses

Set $\log_+(u) := \log_+(1 \vee u)$.

(H1) b is continuous, and there exist two nonnegative constants c_1 and c_2 such that for any $u \in \mathbb{R}$,

$$|b(u)| \leq c_1 |u| \log_+ |u| + c_2. \tag{3.2}$$

(H2) There exist nonnegative constants c_3, c_4, c_5, such that for any $u, v \in \mathbb{R}$,

$$\begin{aligned} |b(u) - b(v)| \leq & c_3 |u - v| \log_+ \frac{1}{|u - v|} \\ & + c_4 \log_+(|u| \vee |v|)|u - v| + c_5 |u - v|. \end{aligned} \tag{3.3}$$

Note that condition (H2) implies condition (H1).

Example 3

The function $x \mapsto x \log |x|$ satisfies (H2). For any $x, y \in \mathbb{R}$,

$$
\begin{aligned}
|x \log |x| - y \log |y|| \leq & |x-y| \log \frac{1}{|x-y|} \\
& + [\log_+(|x| \vee |y|) + 1 + \log 2]|x-y|.
\end{aligned}
\tag{3.4}
$$

Main results

Here are the main results.

Theorem 4

Assume $u_0 \in C_{tem}$ and that (H1) is satisfied. If σ is bounded and continuous, then there exists a weak (in the probabilistic sense) solution to the stochastic reaction-diffusion equation (2.1) with sample paths a.s. in $C(\mathbb{R}_+, C_{tem})$.

Main results

Theorem 5

Assume $u_0 \in C_{tem}$ and that (H2) is satisfied. If σ is bounded and Lipchitz, then the pathwise uniqueness holds for solutions of (2.1) in $C(\mathbb{R}_+, C_{tem})$. Hence there exists a unique strong solution to (2.1) in $C(\mathbb{R}_+, C_{tem})$.

Preliminaries

Set $\log_+(r) := \log(r \vee 1)$.

Lemma 6

Let X, a, c_1, c_2 *be nonnegative functions on* $\mathbb{R}_+$, M *an increasing function with* $M(0) \geq 1$. *Moreover, suppose that* c_1, c_2 *be integrable on finite time intervals. Assume that for any* $t \geq 0$,

$$X(t) + a(t) \leq M(t) + \int_0^t c_1(s)X(s)\,\mathrm{d}s + \int_0^t c_2(s)X(s)\log_+ X(s)\,\mathrm{d}s, \quad (4.1)$$

and the above integral is finite. Then for any $t \geq 0$,

$$X(t) + a(t) \leq M(t)^{\exp(C_2(t))}\exp\left(\exp(C_2(t))\int_0^t c_1(s)\exp(-C_2(s))\,\mathrm{d}s\right), \quad (4.2)$$

where $C_2(t) := \int_0^t c_2(s)ds\,\mathrm{d}s$.

Lemma 7

Let $Y(t)$ be a nonnegative function on $\mathbb{R}_+$. Let c_1 and c_2 be non-negative, increasing functions on $\mathbb{R}_+$. Let $\varepsilon \in [0,1)$ be a constant and $c_3 : \mathbb{R}_+ \times (\varepsilon, 1) \longmapsto \mathbb{R}_+$ be a function that is increasing with respect the first variable. Suppose that for any $\theta \in (\varepsilon, 1)$, the following integral inequality holds

$$
\begin{aligned}
Y(t) \quad \leq \quad & c_1(t) \int_0^t Y(s)\,\mathrm{d}s + c_2(t) \int_0^t Y(s) \log_+ \frac{1}{Y(s)}\,\mathrm{d}s \\
& + c_3(t,\theta) \int_0^t Y(s)^\theta\,\mathrm{d}s, \quad \forall\, t \geq 0. \qquad (4.3)
\end{aligned}
$$

Preliminaries

If for any $t > 0$,

$$\limsup_{\theta \to 1-} (1-\theta) c_3(t,\theta) < \infty, \tag{4.4}$$

then $Y(t) = 0$ for any $t \geq 0$. In particular, if $c_3(t,\theta) \leq \frac{c(t)}{1-\theta}$ and c is an increasing function with respect to t, then (4.4) holds.

Sketch of the proof. Fix any $T > 0$ we will show that $Y(\cdot) = 0$ on $[0, T]$. Let

$$\delta_T := \limsup_{\theta \to 1-} (1-\theta) c_3(T, \theta), \qquad T^* := \min\left\{\frac{1}{3\delta_T}, \frac{e}{3c_2(T)}\right\}. \tag{4.5}$$

We first prove $Y(t) = 0$ for $t \in [0, T^* \wedge T]$. Since

$$\sup_{x \geq 0} \left(x \log_+ \frac{1}{x} \right) = \frac{1}{e}, \tag{4.6}$$

we have

$$\begin{aligned} Y(t) \leq & c_1(t) \int_0^t Y(s)\,\mathrm{d}s + \frac{c_2(t)}{1-\theta} \int_0^t Y(s)^\theta Y(s)^{1-\theta} \log_+ \frac{1}{Y(s)^{1-\theta}}\,\mathrm{d}s \\ & + c_3(t,\theta) \int_0^t Y(s)^\theta\,\mathrm{d}s \\ \leq & c_1(t) \int_0^t Y(s)\,\mathrm{d}s + \left[\frac{c_2(t)}{e(1-\theta)} + c_3(t,\theta) \right] \int_0^t Y(s)^\theta\,\mathrm{d}s. \end{aligned} \tag{4.7}$$

For $t \in [0, T]$, let

$$\Phi(t) := c_1(T) \int_0^t Y(s)\,\mathrm{d}s + \left[\frac{c_2(T)}{e(1-\theta)} + c_3(T,\theta)\right] \int_0^t Y(s)^\theta\,\mathrm{d}s. \tag{4.8}$$

Then $Y(t) \le \Phi(t)$ for any $t \in [0, T]$. Thus,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) =& c_1(T) Y(t) + \left[\frac{c_2(T)}{e(1-\theta)} + c_3(T,\theta)\right] Y(t)^\theta \\ \le & c_1(T)\Phi(t) + \left[\frac{c_2(T)}{e(1-\theta)} + c_3(T,\theta)\right] \Phi(t)^\theta. \end{aligned} \tag{4.9}$$

Preliminaries

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi(t)^{1-\theta}\right) \leq (1-\theta)c_1(T)\Phi(t)^{1-\theta} + \left[\frac{c_2(T)}{e} + c_3(T,\theta)(1-\theta)\right]. \tag{4.10}$$

Solving the above inequality, we obtain

$$\Phi(t)^{1-\theta} \leq \left[\frac{c_2(T)}{e} + c_3(T,\theta)(1-\theta)\right]\int_0^t e^{(1-\theta)c_1(T)(t-s)}\,\mathrm{d}s. \tag{4.11}$$

Hence

$$
\begin{aligned}
Y(t) \leq &\Phi(t) \leq \left\{ \left[\frac{c_2(T)T^*}{e} + (1-\theta)c_3(T,\theta)T^* \right] e^{(1-\theta)c_1(T)T^*} \right\}^{\frac{1}{1-\theta}} \\
\leq &e^{c_1(T)T^*} \left\{ \frac{c_2(T)T^*}{e} + (1-\theta)c_3(T,\theta)T^* \right\}^{\frac{1}{1-\theta}}, \qquad (4.12)
\end{aligned}
$$

for any $t \in [0, T^*]$. Letting $\theta \to 1$ and in view of the definition of T^*,

$$
Y(t) = 0, \quad \forall\, t \in [0, T^*]. \qquad (4.13)
$$

Lemma 8

The following estimates of the heat kernel $p_t(x,y)$ hold.

(i) *For any $x, y \in \mathbb{R}$, $\theta \in [0,1]$, $0 < s \le t$,*

$$|p_t(x,y) - p_s(x,y)| \le \frac{(2\sqrt{2})^\theta |t-s|^\theta}{s^\theta} \big(p_s(x,y) + p_t(x,y) + p_{2t}(x,y)\big). \tag{4.14}$$

(ii) *For any $x, y \in \mathbb{R}$ and $t > 0$,*

$$\int_{\mathbb{R}} |p_t(x,z) - p_t(y,z)| \,\mathrm{d}z \le \sqrt{\frac{2}{\pi}} \times \frac{|x-y|}{\sqrt{t}}. \tag{4.15}$$

(iii) *For any $x, y \in \mathbb{R}$ and $\eta, t > 0$,*

$$\int_{\mathbb{R}} |p_t(x,z) - p_t(y,z)| e^{\eta|z|} \,\mathrm{d}z \le 2\sqrt{2} \times \frac{|x-y|}{\sqrt{t}} \times e^{\eta^2 t} \times e^{\eta(|x|+|x-y|)}.$$

Preliminaries

(iv) For any $x, y \in \mathbb{R}$ and $\eta, t > 0$,

$$\int_{\mathbb{R}} |p_t(x,z) - p_t(y,z)| e^{\eta|z|} \eta|z| \,\mathrm{d}z$$
$$\leq \frac{\sqrt{2}|x-y|}{\sqrt{t}} \times \Big[e^{\eta^2 t} \times e^{\eta(|x|+|x-y|)} \eta(|x| + |x-y|)$$
$$+ 2e^{\eta^2 t} \left(2\eta^2 t + \eta\sqrt{\frac{t}{\pi}} \right) e^{\eta(|x|+|x-y|)} \Big]. \tag{4.17}$$

(v) For any $x, y \in \mathbb{R}$ and $0 < s \leq t$,

$$\int_0^s \int_{\mathbb{R}} |p_{t-r}(x,z) - p_{s-r}(y,z)|^2 \,\mathrm{d}r\mathrm{d}z$$
$$\leq \frac{\sqrt{2}-1}{\sqrt{\pi}} |t-s|^{\frac{1}{2}} + \frac{2}{\sqrt{\pi}} |x-y|. \tag{4.18}$$

Moment estimates of stochastic convolutions

Lemma 9

Let $h : \mathbb{R}_+ \longmapsto \mathbb{R}_+$ *be an increasing function. Let* $\{\sigma(s,y) : (s,y) \in \mathbb{R}_+ \times [0,1]\}$ *be a random field such that the following stochastic convolution with respect to space time white noise is well defined. Let* τ *be a stopping time. Then for any* $p > 10$ *and* $T > 0$*, there exists a constant* $C_{p,h(T),T} > 0$ *such that*

$$\mathbb{E} \sup_{(t,x)\in[0,T\wedge\tau]\times\mathbb{R}} \left\{ \left| \int_0^t \int_{\mathbb{R}} p_{t-s}(x,y)\sigma(s,y)\, W(\mathrm{d}s,\mathrm{d}y) \right| e^{-h(t)|x|} \right\}^p$$
$$\leq C_{p,h(T),T}\, \mathbb{E} \int_0^{T\wedge\tau} \int_{\mathbb{R}} |\sigma(t,x)|^p e^{-ph(t)|x|}\, \mathrm{d}x\mathrm{d}t. \tag{5.1}$$

In particular, if σ *is bounded and* h *is a positive constant, then the left hand side of (5.1) is finite.*

We employ the factorization method. Choose α such that $\frac{3}{2p} < \alpha < \frac{1}{4} - \frac{1}{p}$. This is possible because $p > 10$. Let

$$(J_\alpha\sigma)(s, y) := \int_0^s \int_0^1 (s-r)^{-\alpha} p_{s-r}(y, z)\sigma(r, z)W(dr, dz), \tag{5.2}$$

$$(J^{\alpha-1}f)(t, x) := \frac{sin\pi\alpha}{\pi} \int_0^t \int_0^1 (t-s)^{\alpha-1} p_{t-s}(x, y)f(s, y)dsdy. \tag{5.3}$$

Stochastic Fubini theorem $\Rightarrow$ for any $(t, x) \in \mathbb{R}_+ \times [0, 1]$,

$$\int_0^t \int_0^1 p_{t-s}(x, y)\sigma(s, y)W(ds, dy) = J^{\alpha-1}(J_a\sigma)(t, x). \tag{5.4}$$

Based on the above identity, using BGD inequalities and Holder inequality etc we can prove (5.1).

Proposition 10

Let $h : \mathbb{R}_+ \longmapsto \mathbb{R}_+$ be an increasing function. Let $\{\sigma(s,y) : (s,y) \in \mathbb{R}_+ \times [0,1]\}$ be a random field such that the following stochastic convolution with respect to the space time white noise is well defined. Let τ be a stopping time. Then for any $\epsilon, T > 0$, and $0 < p \leq 10$, there exists a constant $C_{\epsilon,p,h(T),T}$ such that

$$\begin{aligned}
&\mathbb{E} \sup_{(t,x)\in[0,T\wedge\tau]\times\mathbb{R}} \left\{ \left| \int_0^t \int_{\mathbb{R}} p_{t-s}(x,y)\sigma(s,y)\, W(\mathrm{d}s,\mathrm{d}y) \right| e^{-h(t)|x|} \right\}^p \\
\leq &\epsilon\, \mathbb{E} \sup_{(t,x)\in[0,T\wedge\tau]\times\mathbb{R}} \left(|\sigma(t,x)| e^{-h(t)|x|} \right)^p \\
&+ C_{\epsilon,p,h(T),T}\, \mathbb{E} \int_0^{T\wedge\tau} \int_{\mathbb{R}} |\sigma(t,x)|^p e^{-ph(t)|x|} \,\mathrm{d}x\mathrm{d}t. \qquad (5.5)
\end{aligned}$$

Proof. The crucial step is to prove the following tail estimate.

Claim: For any ρ, $T > 0$ and $q > 10$,

$$\begin{aligned}&\mathbb{P}\left(\sup_{(t,x)\in[0,T\wedge\tau]\times\mathbb{R}}\left[\left|\int_0^t\int_{\mathbb{R}} p_{t-s}(x,y)\sigma(s,y)\,W(\mathrm{d}s,\mathrm{d}y)\right| e^{-h(t)|x|}\right] > \rho\right)\\ \leq&\mathbb{P}\left(\int_0^{T\wedge\tau}\int_{\mathbb{R}}|\sigma(s,y)|^q\, e^{-qh(s)|y|}\,\mathrm{d}y\mathrm{d}s > \rho^q\right)\\ &+\frac{C_{q,h(T),T}}{\rho^q}\mathbb{E}\min\left\{\rho^q,\int_0^{T\wedge\tau}\int_{\mathbb{R}}|\sigma(s,y)|^q\, e^{-qh(t)|y|}\,\mathrm{d}y\mathrm{d}s\right\}.\end{aligned} \tag{5.6}$$

A priori estimate of solutions

For $\lambda, \kappa > 0$, set

$$\beta(\lambda,\kappa) := \max\left\{\frac{\lambda^2}{2}, 4\kappa\right\}, \tag{5.7}$$

$$T^*(\lambda,\kappa) := \frac{1}{2\beta(\lambda,\kappa)}\left[1 + \log\left(\frac{4\beta(\lambda,\kappa)}{\lambda^2}\log\frac{\beta(\lambda,\kappa)}{2\kappa}\right)\right], \tag{5.8}$$

$$V(t,x) := \int_0^t \int_{\mathbb{R}} p_{t-s}(x,y)\sigma(u(s,y))\, W(\mathrm{d}s,\mathrm{d}y). \tag{5.9}$$

It is easy to see that for any $\kappa > 0$, $T^*(\lambda,\kappa) \to \infty$ as $\lambda \to 0$.

Lemma 11

Assume that (H1) is satisfied and σ is bounded. Let u be a solution of (2.1). Then for any $\lambda > 0$ and $T \leq T^*(\lambda, c_1)$, there exists a constant $C_{\lambda,c_1,T}$ such that the following a priori estimate holds for $\mathbb{P}$-a.s.,

$$
\begin{aligned}
&\sup_{t\le T, x\in\mathbb{R}} \left(|u(t,x)| e^{-\lambda |x| e^{\beta t}} \right) \\
\le & C_{\lambda, c_1, T} \times \Bigg\{ 1 + 2c_2 T + 4 e^{\frac{\lambda^2 T}{2}} \sup_{x\in\mathbb{R}} \left(|u_0(x)| e^{-\lambda |x|} \right) \\
& + 2 \sup_{(t,x)\in[0,T]\times\mathbb{R}} \left(|V(t,x)| e^{-\lambda |x|} \right) \Bigg\}^{e^{4c_1 T e^{\frac{\lambda^2}{4\beta} e^{2\beta T - 1}}}}, \qquad (5.10)
\end{aligned}
$$

where we write β instead of $\beta(\lambda, c_1)$ for simplicity, and the constant c_1 is same as that in condition (H1).

Remark 12

Lemma 11 actually implies that the solutions of (2.1) don't blow up in the space C_{tem}, since we can take sufficiently small $\lambda > 0$ such that $T^*(\lambda, c_1)$ can be larger than any given number.

A brief sketch of the proof. Set

$$U(T) := \sup_{(t,x)\in[0,T]\times\mathbb{R}} \left(|u(t,x)| e^{-\lambda |x| e^{\beta t}} \right).$$

From (3.1), we have

$$\begin{aligned} U(T) \leq & \sup_{t\leq T, x\in\mathbb{R}} \left(|P_t u_0(x)| e^{-\lambda |x| e^{\beta t}} \right) + \sup_{t\leq T, x\in\mathbb{R}} \left(|V(t,x)| e^{-\lambda |x|} \right) \\ & + \sup_{t\leq T, x\in\mathbb{R}} \left\{ \left| \int_0^t \int_{\mathbb{R}} p_{t-s}(x,y) b(u(s,y)) \,\mathrm{d}s\mathrm{d}y \right| e^{-\lambda |x| e^{\beta t}} \right\}. \qquad (5.11) \end{aligned}$$

The difficulty lies in dealing with the superlinear drift. We have

$$
\begin{aligned}
&\sup_{t\le T, x\in\mathbb{R}} \left\{ \left| \int_0^t \int_{\mathbb{R}} p_{t-s}(x,y) b(u(s,y)) \, \mathrm{d}s\mathrm{d}y \right| e^{-\lambda |x| e^{\beta t}} \right\} \\
\le & \sup_{t\le T, x\in\mathbb{R}} \left\{ \int_0^t \int_{\mathbb{R}} p_{t-s}(x,y) \left(c_1 |u(s,y)| \log_+ |u(s,y)| + c_2 \right) \, \mathrm{d}s\mathrm{d}y \times e^{-\lambda |x| e^{\beta t}} \right\} \\
\le & c_2 T + c_1 \sup_{t\le T, x\in\mathbb{R}} \left\{ \int_0^t \sup_{y\in\mathbb{R}} \left[\left(|u(s,y)| e^{-\lambda |y| e^{\beta s}} \right) \times \log_+ \left(|u(s,y)| e^{-\lambda |y| e^{\beta s}} \right) \right] \right. \\
& \qquad \left. \times \int_{\mathbb{R}} p_{t-s}(x,y) e^{\lambda |y| e^{\beta s}} \, \mathrm{d}y\mathrm{d}s \times e^{-\lambda |x| e^{\beta t}} \right\} \\
& + c_1 \sup_{t\le T, x\in\mathbb{R}} \left\{ \int_0^t \sup_{y\in\mathbb{R}} \left(|u(s,y)| e^{-\lambda |y| e^{\beta s}} \right) \right. \\
& \qquad \left. \times \int_{\mathbb{R}} p_{t-s}(x,y) e^{\lambda |y| e^{\beta s}} \lambda |y| e^{\beta s} \, \mathrm{d}y\mathrm{d}s \times e^{-\lambda |x| e^{\beta t}} \right\} =: c_2 T + I + II.
\end{aligned}
\tag{5.12}
$$

Note that the function $x \mapsto x \log_+ x$ is increasing on $[0, \infty)$, so we have

$$\begin{aligned} I \leq & c_1 \sup_{t \leq T, x \in \mathbb{R}} \left\{ \int_0^t \sup_{y \in \mathbb{R}, r \leq s} \left[\left(|u(r,y)| e^{-\lambda |y| e^{\beta r}} \right) \times \log_+ \left(|u(r,y)| e^{-\lambda |y| e^{\beta r}} \right) \right] \right. \\ & \left. \times\, 2 e^{\frac{\lambda^2 (t-s) e^{2\beta s}}{2}} e^{\lambda |x| e^{\beta s}} \, \mathrm{d}s \times e^{-\lambda |x| e^{\beta t}} \right\} \\ \leq & 2 c_1 \sup_{t \leq T} \left\{ \sup_{s \leq t} \left(e^{\frac{\lambda^2 (t-s) e^{2\beta s}}{2}} \right) \int_0^t U(s) \log_+ U(s) \, \mathrm{d}s \right\} \\ \leq & 2 c_1 e^{\frac{\lambda^2}{4\beta} e^{2\beta T - 1}} \int_0^T U(s) \log_+ U(s) \, \mathrm{d}s, \end{aligned} \tag{5.13}$$

where we have used the fact that

$$\max_{s \in [0,t]} e^{\frac{\lambda^2 (t-s) e^{2\beta s}}{2}} = e^{\frac{\lambda^2}{4\beta} e^{2\beta t - 1}}. \tag{5.14}$$

For the term II, we estimate as follows

$$
\begin{aligned}
II \leq & c_1 \sup_{t\leq T, x\in\mathbb{R}} \Big\{ \int_0^t \sup_{y\in\mathbb{R}} \Big(|u(s,y)| e^{-\lambda |y| e^{\beta s}} \Big) \\
& \times \Big(e^{\frac{\lambda^2 (t-s) e^{2\beta s}}{2}} e^{\lambda |x| e^{\beta s}} \lambda |x| e^{\beta s} + C_{\lambda,\beta,t} e^{\lambda |x| e^{\beta s}} \Big) \mathrm{d}s \times e^{-\lambda |x| e^{\beta t}} \Big\} \\
\leq & c_1 \sup_{t\leq T, x\in\mathbb{R}} \Big\{ \sup_{s\leq t, y\in\mathbb{R}} \Big(|u(s,y)| e^{-\lambda |y| e^{\beta s}} \Big) \times \frac{1}{\beta} \sup_{s\leq t} \Big(e^{\frac{\lambda^2 (t-s) e^{2\beta s}}{2}} \Big) \\
& \times \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} e^{\lambda |x| e^{\beta s}} \mathrm{d}s \times e^{-\lambda |x| e^{\beta t}} \Big\} \\
& + c_1 \sup_{t\leq T} \Big\{ C_{\lambda,\beta,t} \int_0^t \sup_{r\leq s, y\in\mathbb{R}} \Big(|u(r,y)| e^{-\lambda |y| e^{\beta r}} \Big) \mathrm{d}s \Big\} \\
\leq & \frac{c_1}{\beta} e^{\frac{\lambda^2}{4\beta}} e^{2\beta T - 1} U(T) + C_{\lambda, c_1, T} \int_0^T U(s)\, \mathrm{d}s,
\end{aligned} \tag{5.15}
$$

$$\frac{c_1}{\beta} e^{\frac{\lambda^2}{4\beta}} e^{2\beta T - 1} \leq \frac{1}{2} \iff T \leq T^*(\lambda, c_1) = \frac{1}{2\beta}\left[1 + \log\left(\frac{4\beta}{\lambda^2} \log \frac{\beta}{2c_1}\right)\right]. \tag{5.16}$$

Hence for $T \leq T^*(\lambda, c_1)$,

$$II \leq \frac{1}{2} U(T) + c_1 C_{\lambda,\beta,T} \int_0^T U(s)\,\mathrm{d}s. \tag{5.17}$$

Combining (5.11) - (5.13) and (5.17) together, $\Longrightarrow$ $T \leq T^*(\lambda, c_1)$,

$$\begin{aligned} U(T) \leq & 2e^{\frac{\lambda^2 T}{2}} \sup_{y\in\mathbb{R}} \left(|u_0(y)| e^{-\lambda|y|}\right) + \sup_{t\leq T, x\in\mathbb{R}} \left(|V(t,x)| e^{-\lambda|x|}\right) \\ & + c_2 T + 2c_1 e^{\frac{\lambda^2}{4\beta}} e^{2\beta T - 1} \int_0^T U(s) \log_+ U(s)\,\mathrm{d}s \\ & + \frac{1}{2} U(T) + C_{\lambda,c_1,T} \int_0^T U(s)\,\mathrm{d}s. \end{aligned} \tag{5.18}$$

applying the log Gronwall inequality, (5.10) is deduced.

Existence of weak solutions

Let φ: nonnegative smooth function, *supp* $\varphi \subset (-1,1)$ and $\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x = 1$. Let $\{\eta_n\}_{n\geq 1}$: cut-off functions, $0 \leq \eta_n \leq 1$, $\eta_n(x) = 1$ if $|x| \leq n$, and $\eta_n(x) = 0$ if $|x| \geq n+2$. Define

$$b_n(x) := n\int_{\mathbb{R}} b(y)\varphi(n(x-y))\,\mathrm{d}y \times \eta_n(x), \tag{6.1}$$

$$\sigma_n(x) := n\int_{\mathbb{R}} \sigma(y)\varphi(n(x-y))\,\mathrm{d}y \times \eta_n(x). \tag{6.2}$$

Consider the approximating SPDEs:

$$\begin{aligned} u_n(t,x) =& P_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x,y) b_n(u_n(s,y))\,\mathrm{d}s\mathrm{d}y \\ &+ \int_0^t \int_{\mathbb{R}} p_{t-s}\sigma_n(u_n(s,y))\, W(\mathrm{d}s,\mathrm{d}y). \end{aligned} \tag{6.3}$$

It is known that there exists a unique solution u_n to the above equation. Moreover, the sample paths of u_n are a.s. in $C(\mathbb{R}_+, C_{tem})$. The following result is a uniform bound for the solutions u_n.

Lemma 13

Assume $u_0 \in C_{tem}$ and (H1). Suppose that σ is bounded and continuous. Then for any $p \geq 1$ and $\lambda, T > 0$, we have

$$\sup_{n\geq 1} \mathbb{E}\left[\sup_{t\leq T, x\in\mathbb{R}} \left(|u_n(t,x)|e^{-\lambda|x|}\right)^p\right] < \infty. \tag{6.4}$$

Existence of weak solutions

Define

$$X_n(t,x) := \int_0^t \int_{\mathbb{R}} p_{t-r}(x,z) b_n(u_n(r,z))\, \mathrm{d}r \mathrm{d}z, \quad n \geq 1. \tag{6.5}$$

$$V_n(t,x) := \int_0^t \int_{\mathbb{R}} p_{t-s} \sigma_n(u_n(s,y))\, W(\mathrm{d}s, \mathrm{d}y) \tag{6.6}$$

To get the tightness of the approximating solutions $\{u_n\}$ we need to prove the following result.

Existence of weak solutions

Lemma 14

Let $u_0 \in C_{tem}$. Assume that (H1) holds and that σ is continuous with $K_\sigma := \sup_{z\in\mathbb{R}} |\sigma(z)| < \infty$. Then for any $\lambda, T > 0$, $p \geq 1$ and $\theta \in (0,1)$, there exist constants $C_{\lambda,c_1,L_b,K_\sigma,T,p,\theta,u_0}$ and $C_{K_\sigma,T,p}$ independent of n such that

$$\mathbb{E}\left(|X_n(t,x) - X_n(s,y)|^p e^{-\lambda|x|}\right) \leq C_{\lambda,c_1,L_b,K_\sigma,T,p,\theta,u_0}\left(|t-s|^{\theta p} + |x-y|^p\right), \tag{6.7}$$

$$\mathbb{E}\left(|V_n(t,x) - V_n(s,y)|^p e^{-\lambda|x|}\right) \leq C_{K_\sigma,T,p}\left(|t-s|^{\frac{p}{4}} + |x-y|^{\frac{p}{2}}\right), \tag{6.8}$$

for any $s,t \in [0,T]$ and $x,y \in \mathbb{R}$ with $|x-y| \leq 1$. In particular, the family $\{u_n\}$ is tight in $C(\mathbb{R}_+, C_{tem})$.

Existence of weak solutions

Since $\{u_n\}$ is tight in $C(\mathbb{R}_+, C_{tem})$. By Prokhorov's theorem and Skorokhod's representation theorem, we may assume that $d(u_n, u) \to 0$ (not relabelled) a.s. in $C(\mathbb{R}_+, C_{tem})$ for some process u on some probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, in other words, for any $\lambda > 0$, $T \geq 0$,

$$\sup_{t \leq T, x \in \mathbb{R}} \left(|u_n(t,x) - u(t,x)| e^{-\lambda |x|} \right) \to 0, \quad \widetilde{\mathbb{P}} - a.s.. \tag{6.9}$$

It follows that for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\int_0^t \int_{\mathbb{R}} p_{t-r}(x,z) b_n(u_n(r,z)) \, \mathrm{d}r \mathrm{d}z \to \int_0^t \int_{\mathbb{R}} p_{t-r}(x,z) b(u(r,z)) \, \mathrm{d}r \mathrm{d}z,$$

$$\int_0^t \int_{\mathbb{R}} p_{t-r}(x,z) \sigma_n(u_n(r,z)) \, W(\mathrm{d}r, \mathrm{d}z) \to \int_0^t \int_{\mathbb{R}} p_{t-r}(x,z) \sigma(u(r,z)) \, W(\mathrm{d}r, \mathrm{d}z), \tag{6.10}$$

as $n \to \infty$. Therefore, we see that u is a weak solution of (2.1).

Pathwise uniqueness

Suppose that u, v are two solutions of equation (2.1), $u, v \in C(\mathbb{R}_+, C_{tem})$. Fix $T > 0$, and take $\lambda > 0$ sufficiently small so that $T \leq T^*(\lambda, c_4)$. In this section, we write β for $\beta(\lambda, c_4)$ for simplicity. Let $M > 0$ and $0 < \delta \leq e^{-1}$.

$$\begin{aligned}\tau_M :=& \inf\left\{t > 0 : \sup_{x\in\mathbb{R}}\left(|u(t,x)|e^{-\lambda|x|e^{\beta t}}\right) \geq M\right\} \\ & \wedge \inf\left\{t > 0 : \sup_{x\in\mathbb{R}}\left(|v(t,x)|e^{-\lambda|x|e^{\beta t}}\right) \geq M\right\}, \\ \tau^\delta :=& \inf\left\{t > 0 : \sup_{x\in\mathbb{R}}\left(|u(t,x) - v(t,x)|e^{-\lambda|x|e^{\beta t}}\right) \geq \delta\right\}, \\ \tau_M^\delta :=& \tau_M \wedge \tau^\delta \wedge T,\end{aligned}$$

with the convention that $\inf\emptyset = +\infty$.

$$Z(r) := \mathbb{E} \sup_{t\leq r\wedge\tau_M^\delta, x\in\mathbb{R}} \left(|u(t,x) - v(t,x)|e^{-\lambda|x|e^{\beta t}}\right). \tag{7.1}$$

$$
\begin{aligned}
& Z(r) \\
\leq\, & \mathbb{E} \sup_{t \leq r \wedge \tau_M^\delta, x \in \mathbb{R}} \Big\{ \int_0^t \int_{\mathbb{R}} p_{t-s}(x,y) |b(u(s,y)) - b(v(s,y))| \, \mathrm{d}s \mathrm{d}y \\
& \qquad \times e^{-\lambda |x| e^{\beta t}} \Big\} \\
& + \mathbb{E} \sup_{t \leq r \wedge \tau_M^\delta, x \in \mathbb{R}} \Big\{ \Big| \int_0^t \int_{\mathbb{R}} p_{t-s}(x,y) [\sigma(u(s,y)) - \sigma(v(s,y))] \\
& \qquad W(\mathrm{d}s, \mathrm{d}y) \Big| e^{-\lambda |x| e^{\beta t}} \Big\} \\
=:\, & I + J.
\end{aligned} \tag{7.2}
$$

$$
\begin{aligned}
I \leq \mathbb{E} \sup_{t \leq r \wedge \tau_M^\delta, x \in \mathbb{R}} \Big\{ \int_0^t \int_{\mathbb{R}} & p_{t-s}(x,y) c_3 |u(s,y) - v(s,y)| \\
& \times \log_+ \frac{1}{|u(s,y) - v(s,y)|} \mathrm{d}s\mathrm{d}y \times e^{-\lambda |x| e^{\beta t}} \Big\} \\
+ \mathbb{E} \sup_{t \leq r \wedge \tau_M^\delta, x \in \mathbb{R}} \Big\{ \int_0^t \int_{\mathbb{R}} & p_{t-s}(x,y) c_4 \log_+ \left(|u(s,y)| \vee |v(s,y)| \right) \\
& \times |u(s,y) - v(s,y)| \, \mathrm{d}s\mathrm{d}y \times e^{-\lambda |x| e^{\beta t}} \Big\} \\
+ \mathbb{E} \sup_{t \leq r \wedge \tau_M^\delta, x \in \mathbb{R}} \Big\{ \int_0^t \int_{\mathbb{R}} & p_{t-s}(x,y) c_5 |u(s,y) - v(s,y)| \, \mathrm{d}s\mathrm{d}y \\
& \times e^{-\lambda |x| e^{\beta t}} \Big\} \\
=: I_1 + I_2 + I_3. & \qquad (7.3)
\end{aligned}
$$

$$
\begin{aligned}
I_1 \leq & \cdots \\
\leq & 2c_3 e^{\frac{\lambda^2}{4\beta} e^{2\beta r - 1}} \int_0^r Z(s) \log_+ \frac{1}{Z(s)} \,\mathrm{d}s,
\end{aligned} \tag{7.4}
$$

$$
\begin{aligned}
I_2 \leq & \cdots \\
\leq & \frac{1}{2} Z(r) + c_4 C_{\lambda,\beta,M,r} \int_0^r Z(s) \,\mathrm{d}s,
\end{aligned} \tag{7.5}
$$

Similarly,

$$
I_3 \leq 2c_5 e^{\frac{\lambda^2}{4\beta} e^{2\beta r - 1}} \int_0^r Z(s) \,\mathrm{d}s. \tag{7.6}
$$

$$J \le \epsilon\, \mathbb{E} \sup_{s \le r\wedge\tau_M^\delta, y\in\mathbb{R}} \left(|\sigma(u(s,y)) - \sigma(v(s,y))| e^{-\lambda|y|e^{\beta s}} \right)$$

$$+ C_{\epsilon,\lambda,\beta,r}\, \mathbb{E} \int_0^{r\wedge\tau_M^\delta} \int_{\mathbb{R}} |\sigma(u(s,y)) - \sigma(v(s,y))| e^{-\lambda|y|e^{\beta s}} \,\mathrm{d}y\mathrm{d}s,$$

Hence for any $0 < \theta < 1$, we have

$$\begin{aligned} J \le & \epsilon L_\sigma Z(r) + C_{\epsilon,\lambda,\beta,r}\, \mathbb{E} \int_0^{r\wedge\tau_M^\delta} \sup_{y\in\mathbb{R}} \Big\{ \left(|\sigma(u(s,y)) - \sigma(v(s,y))| e^{-\lambda|y|e^{\beta s}} \right)^\theta \\ & \times \int_{\mathbb{R}} |\sigma(u(s,y)) - \sigma(v(s,y))|^{1-\theta} e^{-(1-\theta)\lambda|y|e^{\beta s}} \,\mathrm{d}y \Big\} \,\mathrm{d}s \\ \le & \epsilon L_\sigma Z(r) + \frac{(2K_\sigma)^{1-\theta} L_\sigma^\theta C_{\epsilon,\lambda,\beta,r}}{(1-\theta)\lambda} \int_0^r Z(s)^\theta \,\mathrm{d}s. \end{aligned} \tag{7.7}$$

Combining (7.2)-(7.7) together, we obtain that

$$
\begin{aligned}
Z(r) \leq \left(\frac{1}{2} + \epsilon L_\sigma\right) Z(r) + C_{\lambda,M,c_4,c_5,r} \int_0^r Z(s)\,\mathrm{d}s \\
+ 2c_3 e^{\frac{\lambda^2}{4\beta}} e^{2\beta r-1} \int_0^r Z(s) \log_+ \frac{1}{Z(s)}\,\mathrm{d}s + \frac{(2K_\sigma)^{1-\theta} L_\sigma^\theta C_{\epsilon,\lambda,\beta,r}}{(1-\theta)\lambda} \int_0^r Z(s)^\theta\,\mathrm{d}s.
\end{aligned}
\tag{7.8}
$$

Taking for example $\epsilon = \frac{1}{4L_\sigma}$, and then applying the special Gronwall-type inequality established in Lemma 7, we obtain

$$
Z(r) \equiv 0, \quad \forall\, r \geq 0. \tag{7.9}
$$

This further implies that $\tau^\delta \geq T$, $\mathbb{P}$-a.s., otherwise it contradicts the definition of τ^δ. By the arbitrariness of T, we obtain that for $\mathbb{P}$-a.s.,

$$
u(t,x) = v(t,x), \quad \forall\, (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{7.10}
$$

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THANK YOU!